# Team Math Test

November 19, 2011

1. Given a right triangle $ABC$ with altitude $CD$ ($D$ on $\overline{AB}$) and right angle at $C$. Length $AC = 15$ and $CD = 12$. What is the area of triangle $ABC$?

2. What are the last 2 digits of $2011^{2011!}$?

3. You have 2 identical green marbles and 2 identical blue marbles that you want to put in 5 different baskets. How many ways can you do this if the baskets are indistinguishable?

4. The sum of the first $n$ triangular numbers is 445. What is $n$? (The $k^{\text{th}}$ triangular number is $\frac{k(k+1)}{2}$.)

5. Find $n$ if $x$, $y$, and $z$ are integers that satisfy

$$\begin{aligned} \log x + \log y &= 1 \\ z \log y + 1 &= \log 1250 \\ y \log x + x \log y + \log z &= \log n. \end{aligned}$$

6. Find the number of ordered pairs $(a, b)$ such that $a$ and $b$ are coprime positive integers and $a + b = 720$. (Two integers $x$ and $y$ are *coprime* if and only if $\gcd(x, y) = 1$.)

7. A mouse walks on a 5 by 5 square tile floor from the upper-left corner tile to the lower-right corner tile. He moves only right or down, from one tile to an immediately adjacent tile. The central tile contains a soft kitty that will certainly kill the mouse if he steps there. How many paths can the mouse take to reach his destination alive?

8. $A$ and $B$ are two points on a circle with center $O$, and $C$ lies outside the circle, on ray $\overrightarrow{AB}$. Given $AB = 24$, $BC = 28$, and $OA = 15$ find $OC$.

9. Compute the remainder of $17^{2011}$ when it is divided by 12.

10. Find the sum of angles $\theta$, $0^\circ \le \theta \le 180^\circ$ that satisfy $\cos(3\theta) + \cos(2\theta) + \cos(\theta) = 0$.

11. 18 people stand on the sides of an equilateral triangle, in 18 spots equally spaced along the triangle. 7 *indistinguishable* septuplets are named Kevin. There is a person on each corner, with at least one Kevin at a corner. How many unique arrangements of Kevins are there? (Two arrangements are identical if one can be rotated to match the other.)

12. What is the greatest common denominator of 1,325,040 and 392,948?

13. $\mathbf{v}$ is a vector defined by $\mathbf{v} = \langle 8, -2x, 3y\rangle \times \langle y, 4, -x\rangle$, where $x$ and $y$ are nonzero real numbers. There exists a number $k$ such that if $k = \mathbf{v} \cdot \langle 1, 2, 0\rangle$, $x$ and $y$ are uniquely determined. Find $k$.

14. Compute

$$\int_0^{3/2} x \sin^2\left(\pi x^2\right) \mathrm{d}x.$$

15. $ABCD$ is a convex quadrilateral, $K$, $L$, $M$, and $N$ are the midpoints of $\overline{CD}$, $\overline{AD}$, $\overline{AB}$, $\overline{BC}$ respectively. $PQRS$ is the quadrilateral formed by the intersections of $\overline{AK}$, $\overline{BL}$, $\overline{CM}$, and $\overline{DN}$. Determine the area of quadrilateral $PQRS$ if the area of quadrilateral $ABCD$ is 3000, and the areas of quadrilaterals $AMQP$ and $CKSR$ are 513 and 388, respectively.

16. If

$$\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}}+\sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}}+\ldots+\sqrt{1+\frac{1}{2010^2}+\frac{1}{2011^2}} = \frac{(p_1)^a(p_2)^b(p_3)^c(p_4)^d(p_5)^e}{(p_6)^f}$$

where $p_1, p_2, p_3, p_4, p_5, p_6$ are distinct primes and $a, b, c, d, e, f$ are positive integers, evaluate $p_1 + p_2 + p_3 + p_4 + p_5 + p_6$.

17. Find $\sum_{n=0}^{10} f^{(n)}(0)$, where $f^{(n)}(x)$ is the $n$th derivative of $f(x)$ with respect to $x$, if $f(x) = e^x \cos(x)$.

18. Let $n$ be a positive integer greater than 5. If $n_1$ and $n_2$ are the two smallest $n$ satisfying

$$(2n+1)^n \equiv 3 \pmod{5},$$

compute $n_1 + n_2$.

19. Let $x, y, z$ be real numbers such that $x^2 + y^2 + z^2 = 2$ and $3xy + 4z$ is minimized. Find $xyz$.

20. Two circles of radius 9 and radius 16 are on the same side of a line. The circles are tangent to the line and externally tangent to each other. A third circle of radius $r$ is tangent to both circles and the line. Find all possible $r$.